Int J Theor Phys (2010) 49: 1405–1410
DOI 10.1007/s10773-010-0320-7

Investigations on the Fermions Tunneling Radiation from the Charged Rotating Kaluza-Klein Spacetime

De-Jiang Qi

Received: 18 January 2010 / Accepted: 11 March 2010 / Published online: 24 March 2010

Abstract The study of Hawking radiation of fermions via tunneling is a hot spot of current topics in black hole physics. By constructing a set of appropriate matrices γ^{μ} for general covariant Dirac equation, the tunneling effect of Kaluza-Klein spacetime was deeply studied. For spacetimes with different dimensions, constructing a set of appropriate γ^{μ} matrices for general covariant Dirac equation is an important technique for fermions tunneling method. As a result, the tunneling probability of Dirac particles and the expected Hawking temperature of the spacetime were successfully recovered.

Keywords Kaluza-Klein spacetime · Fermions tunneling · Dirac equation · Hawking temperature

1 Introduction

The study of quantum effect (Hawking radiation) of black holes is one of the most important contents in modern theoretical physics. In recent years, there has been increasing interest in Hawking radiation of black holes by using a semi-classical quantum tunneling method firstly proposed by Kraus and Wilczek [1] and then elaborated by Parikh and Wilczek [2, 3]. After that, Zhang and Zhao have extended this method from spherically symmetric black holes to general axi-symmetric black holes and even to the cases of the massive and charged particle's tunneling with considerable success [4–6]. There has been considerable work on tunneling radiation of scalar particles from typical black holes [7–13].

Recently, Kerner and Mann have succeeded in applying fermions tunneling from the Rindler space-time and a general non-rotating black hole [14, 15]. They treated the Hawking radiation as a quantum tunneling process, and the Hawking temperature and tunneling rate can be obtained by the fermions tunneling method. This method has been extended to different typical black holes, and the expected Hawking temperatures were obtained, which strengthen the validity and power of the method [16–20]. These work usually involved 3- or 4-dimensional spacetimes. But, in 5-dimensional spacetimes, it has seldom

D.-J. Qi (✉)
Shenyang Institute of Engineering, Shenyang 110136, People's Republic of China
e-mail: qidejiang0504@126.com

been investigated expect in Refs. [21, 22]. Although fermions tunneling radiation from 5-dimensional black holes has been considered in Refs. [21] and [22], the general complicated 5-dimensional charged black holes has not yet been involved.

The aim of the present paper is to extend fermions tunneling method to the charged Kaluza-Klein black hole in five dimension by constructing a set of appropriate matrices γ^μ for general covariant Dirac equation. This case is quite complicated and is also universal in most of 5-dimensional charged black holes. It is expected that our calculation and result of the Letter can strengthen the validity and power of the method.

2 Tunneling Probability and Hawking Temperature

A new charged, rotating Kaluza-Klein black hole solution to the five-dimensional Einstein-Maxwell theory with a Chern-Simon term is constructed by [23]. The metric and the gauge potential of the solution are given by

$$\mathrm{d}s^2 = -\frac{W(r)}{h(r)}\mathrm{d}t^2 + K^2(r)\frac{\mathrm{d}r^2}{W(r)} + \frac{r^2}{4}\{K(r)(\sigma_1^2+\sigma_2^2) + h(r)[f(r)\mathrm{d}t + \sigma_3]^2\} \tag{1}$$

and

$$A = \frac{\sqrt{3}q}{2r^2}\left(\mathrm{d}t - \frac{a}{2}\sigma_3\right), \tag{2}$$

respectively, where the metric functions $W(r), h(r), f(r)$ and $K(r)$ are defined as

$$W(r) = \frac{(r^2+q)^2 - 2(m+q)(r^2-a^2)}{r^4}, \tag{3}$$

$$h(r) = 1 - \frac{a^2q^2}{r^6} + \frac{2a^2(m+q)}{r^4}, \tag{4}$$

$$f(r) = 1 - \frac{2a}{r^2h(r)}\left(\frac{2m+q}{r^2} - \frac{q^2}{r^4}\right), \tag{5}$$

$$K(r) = \frac{(r_\infty^2+q)^2 - 2(m+q)(r_\infty^2-a^2)}{(r_\infty^2-r^2)^2}, \tag{6}$$

and the left-invariant 1-forms on S^3 are given by

$$\sigma_1 = \cos\varphi\mathrm{d}\theta + \sin\varphi\sin\theta\mathrm{d}\phi, \tag{7}$$

$$\sigma_2 = -\sin\varphi\mathrm{d}\theta + \cos\varphi\sin\theta\mathrm{d}\phi, \tag{8}$$

$$\sigma_3 = \mathrm{d}\varphi + \cos\theta\mathrm{d}\phi. \tag{9}$$

For some purpose, we can rewrite the following metric

$$\begin{aligned}\mathrm{d}s^2 = &-\left[\frac{W(r)}{h(r)} - \frac{r^2}{4}f^2(r)h(r)\right]\mathrm{d}t^2 + \frac{K^2(r)}{W(r)}\mathrm{d}r^2 + \frac{r^2}{4}K(r)\mathrm{d}\theta^2 + \frac{r^2}{4}h(r)\mathrm{d}\varphi^2 \\ &+ 2\frac{r^2h(r)f(r)\cos\theta}{4}\mathrm{d}t\mathrm{d}\phi + \frac{r^2}{4}[K(r)\sin^2\theta + h(r)\cos^2\theta]\mathrm{d}\phi^2 \\ &+ 2\frac{r^2h(r)f(r)}{4}\mathrm{d}t\mathrm{d}\varphi + 2\frac{r^2h(r)\cos\theta}{4}\mathrm{d}\phi\mathrm{d}\varphi,\end{aligned} \tag{10}$$

and

$$A = \frac{\sqrt{3}q}{2r^2}\left(\mathrm{d}t - \frac{a}{2}(\mathrm{d}\varphi + \cos\theta \mathrm{d}\phi)\right). \tag{11}$$

The determinant of the metric is

$$\sqrt{-g} = \frac{2K(r)}{W(r)r\sin\theta} \tag{12}$$

and angular velocities on the horizon are

$$\Omega_{\phi H} = \frac{g_{t\phi}}{g_{\phi\phi}} = \frac{h(r_H)f(r_H)\cos\theta}{K(r_H)\sin^2\theta + h(r_H)\cos^2\theta}, \tag{13}$$

$$\Omega_{\varphi H} = \frac{g_{t\varphi}}{g_{\varphi\varphi}} = -\frac{2a[(2m+q)r_H^2 - q^2]}{r_H^6 - a^2q^2 + 2a^2r_H^2(m+q)}. \tag{14}$$

It is necessary to calculate the inverse of the metric as we will need it in several instances

$$\begin{aligned}
&g^{tt} = -\frac{h(r)}{W(r)}, \qquad g^{rr} = \frac{W(r)}{K^2(r)}, \qquad g^{\theta\theta} = \frac{4}{r^2K(r)}, \qquad g^{\phi\phi} = \frac{4}{r^2K(r)\sin^2\theta},\\
&g^{\varphi\varphi} = \frac{4W(r)[K(r)\sin^2\theta + h(r)\cos^2\theta] - r^2h^2(r)f^2(r)K(r)\sin^2\theta}{r^2h(r)W(r)K(r)\sin^2\theta},\\
&g^{tr} = g^{\phi r} = g^{\varphi r} = g^{t\phi} = 0,\\
&g^{t\varphi} = -\frac{h(r)f(r)}{W(r)}, \qquad g^{\phi\varphi} = -\frac{4\cos\theta}{r^2K(r)\sin^2\theta}.
\end{aligned} \tag{15}$$

The motion equation of $1/2$ spin charged Dirac particles in the electromagnetic field can be written as

$$i\gamma^\mu\left(D_\mu + \frac{ie}{\hbar}A_\mu\right)\Psi + \frac{m}{\hbar}\Psi = 0, \tag{16}$$

where

$$D_\mu = \partial_\mu + \frac{i}{2}\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta], \tag{17}$$

and the γ^μ matrices satisfy

$$\{\gamma^\mu, \gamma^\upsilon\} = 2g^{\mu\upsilon} \times I, \tag{18}$$

m and e are the mass and the electric charge of the emitted particles. Choosing a presentation for them in the following form

$$\begin{aligned}
&\gamma^t = \sqrt{-g^{tt}}\gamma^0, \qquad \gamma^r = \sqrt{g^{rr}}\gamma^3, \qquad \gamma^\theta = \sqrt{g^{\theta\theta}}\gamma^1,\\
&\gamma^\phi = \sqrt{g^{\phi\phi}}\gamma^2, \qquad \gamma^\varphi = F\gamma^0 + G\gamma^2 + H\gamma^4,
\end{aligned} \tag{19}$$

where

$$F = \sqrt{-\frac{(g^{t\varphi})^2}{g^{tt}}}, \qquad G = -\frac{g^{tt}g^{\phi\varphi}}{\sqrt{(-g^{tt})^2g^{\phi\phi}}}, \qquad H = \sqrt{g^{\varphi\varphi} - \frac{(g^{t\varphi})^2}{g^{tt}} - \frac{(-g^{tt}g^{\phi\varphi})^2}{(-g^{tt})^2g^{\phi\phi}}}, \tag{20}$$

and the γ^μ's are

$$\gamma^0=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},\qquad \gamma^1=\begin{pmatrix}0 & \sigma^1\\ \sigma^1 & 0\end{pmatrix},\qquad \gamma^2=\begin{pmatrix}0 & \sigma^2\\ \sigma^2 & 0\end{pmatrix},$$
$$\gamma^3=\begin{pmatrix}0 & \sigma^3\\ \sigma^3 & 0\end{pmatrix},\qquad \gamma^4=\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}, \tag{21}$$

here the σ^i $(i=1,2,3)$ are Pauli Sigma matrices, I is a 2×2 identity matrix and gamma matrices γ^a $(a=0,1,2,3,4)$ obey the anticommutation relations

$$\{\gamma^a,\gamma^b\}=2\eta^{ab} \tag{22}$$

with $\eta^{ab}=\mathrm{diag}(-1,1,1,1,1)$ being the flat (Lorentz) metric tensor. Here, we will only analysis the spin-up case since the final result is the same as the spin-down case as can be presented by using the methods described below. We employ the ansatz for the spin-up spinor field Ψ as following

$$\Psi(t,r,\theta,\phi,\varphi)=\begin{pmatrix}B(t,r,\theta,\phi,\varphi)\\ 0\\ D(t,r,\theta,\phi,\varphi)\\ 0\end{pmatrix}\times\exp\left(\frac{i}{\hbar}I_\uparrow(t,r,\theta,\phi,\varphi)\right). \tag{23}$$

Substituting the above ansatz (23) into the Dirac equation (16), and keeping the prominent term, we can get the following equations

$$D[\sqrt{-g^{tt}}(\partial_t I_\uparrow+eA_t)+\sqrt{g^{rr}}\partial_r I_\uparrow+F(\partial_\varphi I_\uparrow+eA_\varphi)]-B[H(\partial_\varphi I_\uparrow+eA_\varphi)+m]=0, \tag{24a}$$

$$D[\sqrt{g^{\theta\theta}}\partial_\theta I_\uparrow+i\sqrt{g^{\phi\phi}}(\partial_\phi I_\uparrow+eA_\phi)+iG(\partial_\varphi I_\uparrow+eA_\varphi)]=0, \tag{24b}$$

$$-B[\sqrt{-g^{tt}}(\partial_t I_\uparrow+eA_t)-\sqrt{g^{rr}}\partial_r I_\uparrow+F(\partial_\varphi I_\uparrow+eA_\varphi)]+D[H(\partial_\varphi I_\uparrow+eA_\varphi)-m]=0, \tag{24c}$$

$$B[\sqrt{g^{\theta\theta}}\partial_\theta I_\uparrow+i\sqrt{g^{\phi\phi}}(\partial_\phi I_\uparrow+eA_\phi)+iG(\partial_\varphi I_\uparrow+eA_\varphi)]=0, \tag{24d}$$

Because there are three Killing vectors $(\partial/\partial_t)^\mu$, $(\partial/\partial_\phi)^\mu$ and $(\partial/\partial_\varphi)^\mu$ in 5-dimensional charged rotating Kaluza-Klein spacetime, we can employ the following ansatz

$$I_\uparrow=-\omega t+j\phi+L\varphi+R(r,\theta)+K, \tag{25}$$

where ω, j and l are the emitted fermion's energy and angular momentum corresponding to the angles ϕ and φ, and K is a complex constant. Substituting the ansatz (25) into (24a)–(24d), we have

$$D[\sqrt{-g^{tt}}(-\omega+eA_t)+\sqrt{g^{rr}}\partial_r R(r,\theta)+F(L+eA_\varphi)]-B[H(L+eA_\varphi)+m]=0, \tag{26a}$$

$$D[\sqrt{g^{\theta\theta}}\partial_\theta R(r,\theta)+i\sqrt{g^{\phi\phi}}(j+eA_\phi)+iG(L+eA_\varphi)]=0, \tag{26b}$$

$$-B[\sqrt{-g^{tt}}(-\omega+eA_t)-\sqrt{g^{rr}}\partial_r R(r,\theta)+F(L+eA_\varphi)]+D[H(L+eA_\varphi)+m]=0, \tag{26c}$$

$$B[\sqrt{g^{\theta\theta}}\partial_\theta R(r,\theta) + i\sqrt{g^{\phi\phi}}(j + eA_\phi) + iG(L + eA_\varphi)] = 0. \tag{26d}$$

From (26a) and (26c), we can get

$$\partial_r R(r,\theta) = \pm\frac{\sqrt{(\omega - e\Phi_+ - j\Omega_{\phi H} - L\Omega_{\varphi H})^2 - (-g^{tt})^{-1}_{,r}(r - r_+)[H^2(L + eA_\varphi)^2 - m^2]}}{\sqrt{g^{rr}_{,r}(-g^{tt})^{-1}_{,r}(r - r_+)}}, \tag{27}$$

where $r_+ = r_H$, and

$$\Phi_+ = A(r_+,\theta) + \Omega_{\phi H}A_\phi(r_+,\theta) + \Omega_{\varphi H}A_\varphi(r_+,\theta). \tag{28}$$

Near the horizon, $R(r,\theta)$ can be separated as $R(r,\theta) = R(r) + \Theta(\theta)$, which corresponds to $\partial_r R(r,\theta) = \partial_r R(r)$. Since the contribution from $\Theta(\theta)$ is completely the same for both the outgoing and ingoing solutions, the contribution of $\Theta(\theta)$ to the imaginary part of the action is canceled out. Solving for the radial function $R(r)$ yields

$$R_\pm = \pm\pi i\frac{K(r_+)\sqrt{h(r_+)}}{W'(r_+)}. \tag{29}$$

The imaginary part of the action can only be due to the pole at the event horizon or to the imaginary part of K. So when a particle is crossing the event horizon each way, the outgoing and ingoing probabilities are respectively given by

$$\begin{aligned} P_{\text{out}} &= \exp\left(-\frac{2}{\hbar}\,\text{Im}\, I_\uparrow\right) = \exp\left[-\frac{2}{\hbar}(\text{Im}\, R_+ + \text{Im}\, K)\right], \\ P_{\text{in}} &= \exp\left(-\frac{2}{\hbar}\,\text{Im}\, I_\uparrow\right) = \exp\left[-\frac{2}{\hbar}(\text{Im}\, R_- + \text{Im}\, K)\right]. \end{aligned} \tag{30}$$

Then the tunneling probability of fermions crossing from inside to outside the event horizon becomes

$$\Gamma = \frac{P_{\text{out}}}{P_{\text{in}}} = \exp(-4\,\text{Im}\, R_+) = \exp\left[\frac{-4\pi K(r_+)\sqrt{h(r_+)}}{W'(r_+)}\right]. \tag{31}$$

From the tunneling probability, Hawking temperature of rotating Kaluza-Klein black hole can be determined as

$$\begin{aligned} T &= \frac{W'(r_+)}{4\pi K(r_+)\sqrt{h(r_+)}} \\ &= \frac{[(r_+^2 + q) - (m + q)](r_\infty^2 - r_+^2)^2}{\pi[(r_\infty^2 + q)^2 - 2(m + q)(r_\infty^2 - a^2)]\sqrt{r_+^6 - a^2(q^2 - 2r_+^2(m + q))}}. \end{aligned} \tag{32}$$

As a result, Hawking temperature of the charged rotating Kaluza-Klein spacetime by fermions tunneling method is successfully recovered.

3 Conclusion

Although the tunneling methods have been applied to deal with Hawking radiation of black holes with considerable success, most of them are limited to the case of 3- or 4-dimensional space-time. In this paper, by constructing a set of appropriate matrices γ^μ, we have extended Kerner and Mann's fermions tunneling method to the charged rotating Kaluza-Klein black hole. Finally, Hawking temperature of the black hole by fermions method is successfully recovered. As far as I know, Hawking radiation of Dirac particles from the high dimensional axi-symmetric black holes has not been deeply studied. So it is interesting to see if fermions tunneling method is still applicable in such black holes, and how to choose the matrices γ^μ for the covariant Dirac equation of 5-dimensional black hole. We expect our next work can report it.

References

1. Kraus, P., Wilczek, F.: Nucl. Phys. B **437**, 231 (1995)
2. Parikh, M.K., Wiltzek, F.: Phys. Rev. Lett. **85**, 5042 (2000)
3. Parikh, M.K.: Phys. Lett. B **546**, 189 (2002)
4. Zhang, J.Y., Zhao, Z.: Phys. Lett. B **618**, 14 (2005)
5. Zhang, J.Y., Zhao, Z.: J. High Energy Phys. **05**, 10055 (2005)
6. Zhang, J.Y., Zhao, Z.: Nucl. Phys. B **725**, 173 (2005)
7. Ma, Z.Z.: Phys. Lett. B **666**, 376 (2008)
8. Ren, J., Zhao, Z., Gao, C.J.: Chin. Phys. Lett. **22**, 2489 (2005)
9. Jiang, Q.Q., Cai, S.Q., Wu, X.: Phys. Lett. B **647**, 200 (2007)
10. Jiang, Q.Q., Cai, S.Q., Wu, X.: Phys. Rev. D **75**, 064029 (2007)
11. Lin, K., Chen, S.W., Yang, S.Z.: Int. J. Theor. Phys. **47**, 2453 (2008)
12. Lin, K., Yang, S.Z.: Int. J. Theor. Phys. **48**, 2061 (2009)
13. Yang, S.Z., Chen, D.Y.: Int. J. Theor. Phys. **46**, 2923 (2007)
14. Kerner, R., Mann, R.B.: Class. Quantum Gravity **25**, 095014 (2008)
15. Kerner, R., Mann, R.B.: Phys. Lett. B **665**, 277 (2008)
16. Li, R., Ren, J.R.: Phys. Lett. B **661**, 370 (2008)
17. Li, H.L., Yang, S.Z., Zhou, T.J., Lin, R.: Europhys. Lett. **84**, 20003 (2008)
18. Li, H.L., Zhou, T.J., Cai, M.: Astrophys. Space Sci. **318**, 215 (2008)
19. Chen, D.Y., Yang, H.T., Zu, X.T.: Eur. Phys. J. C **56**, 119 (2008)
20. Chen, D.Y., Jiang, Q.Q., Zu, X.T.: Phys. Lett. B **665**, 106 (2008)
21. Jiang, Q.Q.: Phys. Rev. D **78**, 044009 (2008)
22. Jiang, Q.Q.: Phys. Lett. B **666**, 517 (2008)
23. Nakagawa, T., Ishihara, H., Matsuno, K., Tomizawa, S.: arXiv:0801.0164 [hep-th]